Optimal Estimation of Large Toeplitz Covariance Matrices

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Outline

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- **Motivation from Asymptotic Equivalence Theory**
- **Main Results**
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Introduction

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. p-variate Gaussian with an unkown Toeplitz covariance matrix $\Sigma_{p\times p}$,

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{p-2} & \sigma_{p-1} \\ \sigma_1 & \sigma_0 & & & \sigma_{p-2} \\ \vdots & & \ddots & & \vdots \\ \sigma_{p-2} & & & \sigma_0 & \sigma_1 \\ \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_1 & \sigma_0 \end{pmatrix}.$$

Goal: Estimate $\Sigma_{p\times p}$ based on the sample $\{\mathbf{X}_i : 1 \le i \le n\}$.

Introduction – Spectral Density Estimation

The model given by observing

$$\mathbf{X}_1 \sim N\left(0, \Sigma_{p\times p}\right)$$

with $\Sigma_{p\times p}$ Toeplitz is commonly called

Spectral Density Estimation

$\mathbf{X}_1$, a stationary centered Gaussian sequence with spectral density f

where

$$f(t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sigma_m \exp(imt) = \frac{1}{2\pi}[\sigma_0 + 2\sum_{m=1}\sigma_m \cos(mt)],\ t \quad [-\pi, \pi].$$

Here we have $\sigma_{-m} = \sigma_m$.

Remark: there is a one-to-one correspondence between f and $\Sigma_{\infty\times\infty}$.

Introduction – Problem of Interest

We want to understand the minimax risk:

$$\inf_{\hat{\Sigma}} \sup_{\mathcal{F}} \mathbb{E} \quad \hat{\Sigma} - \Sigma \quad ^2$$

where $\cdot$ denotes the spectral norm and is some parameter space for f.

Motivation from Asymptotic Equivalence Theory

Golubev, Nussbaum and Z. (2010, AoS)

The **Spectral Density Estimation** given by observing each $\mathbf{X}_i$ is asymptotically equivalent to the **Gaussian white noise**

$$dy_i(t) = \log f(t)dt + 2\pi^{1=2} p^{-1=2} dW_i(t), \ t \quad [-\pi, \pi]$$

under some assumptions on the unknown f.

For example,

$$(M, \epsilon) = \quad f : \ f(t_1) - f(t_2) \ \leq M \ t_1 - t_2 \quad \text{and } f(t) \geq \epsilon \quad .$$

We need $\alpha > 1/2$ to establish the asymptotic equivalence.

Intuitively, the model

$$\mathbf{X}_i \sim N\left(0, \Sigma_{p\times p}\right), \ i = 1, 2, \ldots, n$$

is asymptotically equivalent to

$$dy\left(t\right) = \log f(t)dt + 2\pi^{1=2}\left(np\right)^{-1=2} dW\left(t\right), \ t \quad [-\pi, \pi]$$

possibly under some strong assumptions on the unknown f .

"Equivalent" Losses

Let $\hat{\Sigma}_{\infty\times\infty}$ be a Toeplitz matrix and $\hat{f}$ be the corresponding spectral density.

We know

$$\left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\| = 2\pi \left\|\hat{f} - f\right\|_{\infty}$$

based on a well known result

$$\Sigma_{\infty\times\infty} \quad = 2\pi \quad f \quad_{\infty}$$

where

$$\Sigma_{\infty\times\infty} \quad = \sup_{\|v\|_2=1} \quad \Sigma_{\infty\times\infty} v \quad_2, \text{ and } \quad f \quad_{\infty} = \sup_{x} \; f(x) \quad .$$

Intuitively

$$\left\|\hat{\Sigma}_{p\times p} - \Sigma_{p\times p}\right\| \qquad \left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\| ?$$

Thus optimal estimation on f may imply optimal estimation on Σ.

Question

Can we show

$$\inf_{\hat{}_p}\ \sup_{p\ F_\alpha} \mathbb{E}\left\|\hat{}\ \mathrm{p}\ \ \mathrm{p} \qquad \mathrm{p}\ \ \mathrm{p}\right\|^2 \qquad \left(\frac{np}{\log(pn)}\right)^{\frac{2\alpha}{2\alpha+1}} ?$$

Remark : Classical result on nonparametric function estimation under the sup norm:

$$\inf_{\hat{}_p}\ \sup_{p\ F_\alpha} \mathbb{E}\left\|\hat{f} \quad f\right\|_1^2 \qquad \left(\frac{np}{\log(pn)}\right)^{\frac{2\alpha}{2\alpha+1}} .$$

Again,

- We don't really have the asymptotic equivalence.
- The following claim is very intuitive

$$\left\|\hat{\Sigma}_{p\times p} - \Sigma_{p\times p}\right\| \qquad \left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\|.$$

Main Results –Lower bound

Show that

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \geq c \left(\frac{np}{\log (pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}$$

for some $c > 0$.

Main Results –Lower bound

A more informative model

Observe $\mathbf{Y}_1 = (\mathbf{X}_1, \mathbf{W}_1)$ with a circulant covariance matrix $\tilde{\Sigma}_{(2p-1)\times(2p-1)}$

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{p-2} & \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_2 & \sigma_1 \\ \sigma_1 & \sigma_0 & & & \sigma_{p-2} & \sigma_{p-1} & & & \sigma_2 \\ \vdots & & \ddots & & \vdots & & \ddots & & \vdots \\ \sigma_{p-2} & & & \sigma_0 & \sigma_1 & & & \sigma_{p-1} & \sigma_{p-2} \\ \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_1 & \sigma_0 & \sigma_2 & \cdots & \sigma_{p-2} & \sigma_{p-1} \\ & & & & \cdots\cdots\cdots\cdots & & & & \end{pmatrix}.$$

Define

$$\omega_j = \frac{2\pi j}{2p-1}, \quad j \leq p-1$$

and where

$$f_p(t) = \frac{1}{2\pi}\left(\sigma_0 + 2\sum_{m=1}^{p-1} \sigma_m \cos(mt)\right).$$

It is well known that the spectral decomposition of $\tilde{\Sigma}_{(2p-1)\times(2p-1)}$ can be described as follows:

$$\tilde{\Sigma}_{(2p-1)\times(2p-1)} = \sum_{|j|\leq p-1} \lambda_j \mathbf{u}_j \mathbf{u}_j'$$

where

$$\lambda_j = f_p(\omega_j), \quad j \leq p-1$$

and the eigenvector $\mathbf{u}_j$ doesn't depend on $\sigma_m : 0 \leq m \leq p-1$.

Main Results –Lower bound

The more informative model is *exactly* equivalent to

$$Z_j = f_p(\omega_j)\,\xi_j, \quad j \leqslant p-1, Var(\xi_j) \asymp 1/n.$$

For this model it is easy to show

$$\sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{f} - f \right\|_\infty^2 \geq c \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}.$$

Main Results –Lower bound

We have

$$\left\|\hat{\Sigma}_{p\times p}-\Sigma_{p\times p}\right\| \geq \sup_{t\in[-\ ;\ ]}\left|(\sigma_0-\hat{\sigma}_0)+2\sum_{m=1}^{p}(1-\frac{m}{p})\left(\hat{\sigma}_m-\sigma_m\right)e^{imt}\right|$$

$$= \sup_{t\in[-\ ;\ ]}\left|\hat{f}\left(t\right)-f\left(t\right)\right| + \text{negligible term}$$

based on a fact

$$\Sigma_{p\times p} \geqslant \sup_{t\in[-\ ;\ ]}\frac{1}{p}\ \Sigma_{p\times p}v_t, v_t = \sup_{t\in[-\ ;\ ]}\left|\sigma_0+2\sum_{m=1}^{p}(1-\frac{m}{p})\sigma_m e^{imt}\right|$$

where $v_t = (e^{it}, e^{i2t}, \cdots, e^{ipt})$. Thus

$$\sup_{\mathcal{F}_\alpha}\mathbb{E}\left\|\hat{\Sigma}_{p\times p}-\Sigma_{p\times p}\right\|^2 \geq c\left(\frac{np}{\log\left(pn\right)}\right)^{-\frac{2\alpha}{2\alpha+1}}.$$

Remark: Need to have some assumptions on (n, p, α) such that the "negligible term" is truly negligible.

Main Results – Upper bound

Show that there is a $\hat{\Sigma}_{p\times p}$ such that

$$\sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \leq C \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}$$

for some $C > 0$.

Main Results – Upper bound

Let $\Sigma_k = [\sigma_m 1_{\{m \leqslant k-1\}}]$ be a banding approximation of $\Sigma_{p\times p}$,, and $\tilde{\Sigma}_k$ be a banding approximation of the sample covariance matrix $\hat{\Sigma}_{p\times p}$. Note that $\mathbb{E}\tilde{\Sigma}_k = \Sigma_k$. Let $\hat{\Sigma}_k$ be a Toeplitz version of $\tilde{\Sigma}_k$ by taking the average of elements along the diagonal.

We have

$$\left\|\hat{\Sigma}_k - \Sigma_p\right\|^2 \leq 2\left\|\hat{\Sigma}_k - \Sigma_k\right\|^2 + 2 \quad \Sigma_k - \Sigma_p \quad ^2 \leq 8\pi^2\left(\quad \hat{f}_k - f_k \quad {}^2_\infty + \quad f_k - f_p \quad {}^2_\infty\right)$$

since

$$\Sigma_k \quad \leqslant 2\pi \quad f_k \quad {}_\infty = \sup_{[- \; ; \;]} \sigma_0 + 2\sum_{m=1}^{k-1} \sigma_m \cos(mt) \; .$$

Main Results – Upper bound

Variance-bias trade-off

Variance part:

$$\mathbb{E}\left\|\hat{f}_k - f_k\right\|_\infty^2 \le C\frac{k}{np}\log(np).$$

Bias part:

$$\left\|f_k - f_p\right\|_\infty^2 \le Ck^{-2\alpha}.$$

Set the optimal k : $k_{optimal} \asymp \left(\frac{np}{\log np}\right)^{\frac{1}{2\alpha+1}}$ which gives

$$\sup_{\mathcal{F}_\alpha}\mathbb{E}\left\|\hat{\Sigma}_{p\times p} - \Sigma_{p\times p}\right\|^2 \le C\left(\frac{np}{\log(pn)}\right)^{-\frac{2\alpha}{2\alpha+1}}$$

Remark: For simplicity we consider only the case $k_{optimal} \le p$.

Main Result

Theorem. The minimax risk of estimating the covariance matrix $\Sigma_{p \times p}$ over the class satisfies

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \asymp \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}} ?$$

under some assumptions on (n, p, α).

Remarks

- Full asymptotic equivalence?
- Sharp asymptotic minimaxity?

Summary

- We studied rate-optimality of Toeplitz matrices estimation.
- Le Cam's theory plays important roles.
- Full asymptotic equivalence and sharp asymptotics remain unknown.